Prove:

(Formula (TrisectingAngle): (1))

$$
\underset{a5,a6,a7,a8,b5,b6,b7,b8,c5,c6,c7,c8,r,x1,x10,x2,x3,x4,x5,x6,x7,x8,x9,y1,y10,y2,y3,y4,y5,y6,y7,y8,y9,\eta 1,\eta 2,\eta 3,\eta 4,\eta 5,\mu}{\forall}
\begin{array}{l}
((c8 = 0) \wedge (a5 * r = 0) \wedge (c5 + \frac{1}{2} * b5 * r = 0) \wedge (-1 * r + x4 = 0) \wedge \\
\quad (x5 = 0) \wedge (-1 * r + x7 = 0) \wedge (x8 = 0) \wedge (b5 * (r + (-1) * x1) + a5 * y1 = 0) \wedge \\
\quad (c5 + \frac{1}{2} * a5 * (r + x1) + \frac{1}{2} * b5 * y1 = 0) \wedge (b6 * (r + (-1) * x2) + a6 * y2 = 0) \wedge \\
\quad (c6 + \frac{1}{2} * a6 * (r + x2) + \frac{1}{2} * b6 * y2 = 0) \wedge (c8 + a8 * x3 + b8 * y3 = 0) \wedge \\
\quad (c5 + a5 * x4 + b5 * y4 = 0) \wedge (-1 * b6 * x5 + a6 * y5 = 0) \wedge (c5 + a5 * x5 + b5 * y5 = 0) \wedge \\
\quad (c6 + \frac{1}{2} * a6 * x5 + \frac{1}{2} * b6 * y5 = 0) \wedge (b7 * (x5 + (-1) * x6) + a7 * (-1 * y5 + y6) = 0) \wedge \\
\quad (c7 + \frac{1}{2} * a7 * (x5 + x6) + \frac{1}{2} * b7 * (y5 + y6) = 0) \wedge (c6 + a6 * x7 + b6 * y7 = 0) \wedge \\
\quad (b7 * (-1 * x10 + x8) + a7 * (y10 + (-1) * y8) = 0) \wedge (c6 + a6 * x8 + b6 * y8 = 0) \wedge \\
\quad (c7 + \frac{1}{2} * a7 * (x10 + x8) + \frac{1}{2} * b7 * (y10 + y8) = 0) \wedge (-1 * b7 * x9 + a7 * y9 = 0) \wedge \\
\quad (c7 + \frac{1}{2} * a7 * x9 + \frac{1}{2} * b7 * y9 = 0) \wedge (-1 + r * \eta 1 = 0) \wedge (-1 + (a5^2 + b5^2) * \eta 2 = 0) \wedge \\
\quad (-1 + (a6^2 + b6^2) * \eta 3 = 0) \wedge (-1 + (a7^2 + b7^2) * \eta 4 = 0) \wedge (-1 + (a8^2 + b8^2) * \eta 5 = 0) \wedge \\
\quad (c8 + a8 * \mu 1 + b8 * \mu 2 = 0) \wedge (b7 * (x5 + (-1) * \mu 1) + a7 * (-1 * y5 + \mu 2) = 0) \wedge \\
\quad (c7 + \frac{1}{2} * a7 * (x5 + \mu 1) + \frac{1}{2} * b7 * (y5 + \mu 2) = 0) \wedge (-1 * b7 * \mu 3 + a7 * \mu 4 = 0) \wedge \\
\quad (c6 + a6 * \mu 3 + b6 * \mu 4 = 0) \wedge (c7 + \frac{1}{2} * a7 * \mu 3 + \frac{1}{2} * b7 * \mu 4 = 0) \Rightarrow \\
\quad (x9^2 * y10 + (-2) * x10 * x9 * y9 + (-1) * y10 * y9^2 = 0) \wedge \\
\quad (x9^3 * y3 + (-3) * x3 * x9^2 * y9 + (-3) * x9 * y3 * y9^2 + x3 * y9^3 = 0))
\end{array}
$$

,

with no assumptions.

Proved.

The Theorem is proved by the Groebner Bases method.

The formula in the scope of the universal quantifier is transformed into an equivalent formula that is a conjunction of disjunctions of equalities and negated equalities. The universal quantifier can then be distributed over the individual parts of the conjunction. By this, we obtain:

Independent proof problems:

(Formula (TrisectingAngle): (1).1)

$$\forall_{a5,a6,a7,a8,b5,b6,b7,b8,c5,c6,c7,c8,r,x1,x10,x2,x3,x4,x5,x6,x7,x8,x9,y1,y10,y2,y3,y4,y5,y6,y7,y8,y9,\eta1,\eta2,\eta3,\eta4,\eta5,\mu}$$

$$\begin{aligned}
&((x9^2 * y10 + (-2) * x10 * x9 * y9 + (-y10 * y9^2) = 0) \lor c8 \neq 0 \lor x5 \neq 0 \lor \\
&\quad x8 \neq 0 \lor -1 + r * \eta1 \neq 0 \lor -1 + a5^2 * \eta2 + b5^2 * \eta2 \neq 0 \lor -1 + a6^2 * \eta3 + b6^2 * \eta3 \neq 0 \lor \\
&\quad -1 + a7^2 * \eta4 + b7^2 * \eta4 \neq 0 \lor -1 + a8^2 * \eta5 + b8^2 * \eta5 \neq 0 \lor c5 + \tfrac{1}{2} * b5 * r \neq 0 \lor \\
&\quad (-r) + x4 \neq 0 \lor (-r) + x7 \neq 0 \lor b5 * r + (-b5 * x1) + a5 * y1 \neq 0 \lor \\
&\quad b6 * r + (-b6 * x2) + a6 * y2 \neq 0 \lor b7 * x5 + (-b7 * x6) + (-a7 * y5) + a7 * y6 \neq 0 \lor \\
&\quad b7 * x5 + (-a7 * y5) + (-b7 * \mu1) + a7 * \mu2 \neq 0 \lor \\
&\quad (-b7 * x10) + b7 * x8 + a7 * y10 + (-a7 * y8) \neq 0 \lor (-b6 * x5) + a6 * y5 \neq 0 \lor \\
&\quad (-b7 * x9) + a7 * y9 \neq 0 \lor (-b7 * \mu3) + a7 * \mu4 \neq 0 \lor c5 + a5 * x4 + b5 * y4 \neq 0 \lor \\
&\quad c5 + a5 * x5 + b5 * y5 \neq 0 \lor c5 + \tfrac{1}{2} * a5 * r + \tfrac{1}{2} * a5 * x1 + \tfrac{1}{2} * b5 * y1 \neq 0 \lor \\
&\quad c6 + a6 * x7 + b6 * y7 \neq 0 \lor c6 + a6 * x8 + b6 * y8 \neq 0 \lor c6 + a6 * \mu3 + b6 * \mu4 \neq 0 \lor \\
&\quad c6 + \tfrac{1}{2} * a6 * x5 + \tfrac{1}{2} * b6 * y5 \neq 0 \lor c6 + \tfrac{1}{2} * a6 * r + \tfrac{1}{2} * a6 * x2 + \tfrac{1}{2} * b6 * y2 \neq 0 \lor \\
&\quad c7 + \tfrac{1}{2} * a7 * x9 + \tfrac{1}{2} * b7 * y9 \neq 0 \lor c7 + \tfrac{1}{2} * a7 * \mu3 + \tfrac{1}{2} * b7 * \mu4 \neq 0 \lor \\
&\quad c7 + \tfrac{1}{2} * a7 * x10 + \tfrac{1}{2} * a7 * x8 + \tfrac{1}{2} * b7 * y10 + \tfrac{1}{2} * b7 * y8 \neq 0 \lor \\
&\quad c7 + \tfrac{1}{2} * a7 * x5 + \tfrac{1}{2} * a7 * x6 + \tfrac{1}{2} * b7 * y5 + \tfrac{1}{2} * b7 * y6 \neq 0 \lor \\
&\quad c7 + \tfrac{1}{2} * a7 * x5 + \tfrac{1}{2} * b7 * y5 + \tfrac{1}{2} * a7 * \mu1 + \tfrac{1}{2} * b7 * \mu2 \neq 0 \lor \\
&\quad c8 + a8 * x3 + b8 * y3 \neq 0 \lor c8 + a8 * \mu1 + b8 * \mu2 \neq 0 \lor a5 * r \neq 0)
\end{aligned}$$

(Formula (TrisectingAngle): (1).2)

$$\forall_{a5,a6,a7,a8,b5,b6,b7,b8,c5,c6,c7,c8,r,x1,x10,x2,x3,x4,x5,x6,x7,x8,x9,y1,y10,y2,y3,y4,y5,y6,y7,y8,y9,\eta1,\eta2,\eta3,\eta4,\eta5,\mu}$$

$$\begin{aligned}
&((x9^3 * y3 + (-3) * x3 * x9^2 * y9 + (-3) * x9 * y3 * y9^2 + x3 * y9^3 = 0) \lor \\
&\quad c8 \neq 0 \lor x5 \neq 0 \lor x8 \neq 0 \lor -1 + r * \eta1 \neq 0 \lor -1 + a5^2 * \eta2 + b5^2 * \eta2 \neq 0 \lor \\
&\quad -1 + a6^2 * \eta3 + b6^2 * \eta3 \neq 0 \lor -1 + a7^2 * \eta4 + b7^2 * \eta4 \neq 0 \lor -1 + a8^2 * \eta5 + b8^2 * \eta5 \neq 0 \lor \\
&\quad c5 + \tfrac{1}{2} * b5 * r \neq 0 \lor (-r) + x4 \neq 0 \lor (-r) + x7 \neq 0 \lor b5 * r + (-b5 * x1) + a5 * y1 \neq 0 \lor \\
&\quad b6 * r + (-b6 * x2) + a6 * y2 \neq 0 \lor b7 * x5 + (-b7 * x6) + (-a7 * y5) + a7 * y6 \neq 0 \lor \\
&\quad b7 * x5 + (-a7 * y5) + (-b7 * \mu1) + a7 * \mu2 \neq 0 \lor \\
&\quad (-b7 * x10) + b7 * x8 + a7 * y10 + (-a7 * y8) \neq 0 \lor (-b6 * x5) + a6 * y5 \neq 0 \lor \\
&\quad (-b7 * x9) + a7 * y9 \neq 0 \lor (-b7 * \mu3) + a7 * \mu4 \neq 0 \lor c5 + a5 * x4 + b5 * y4 \neq 0 \lor \\
&\quad c5 + a5 * x5 + b5 * y5 \neq 0 \lor c5 + \tfrac{1}{2} * a5 * r + \tfrac{1}{2} * a5 * x1 + \tfrac{1}{2} * b5 * y1 \neq 0 \lor \\
&\quad c6 + a6 * x7 + b6 * y7 \neq 0 \lor c6 + a6 * x8 + b6 * y8 \neq 0 \lor c6 + a6 * \mu3 + b6 * \mu4 \neq 0 \lor \\
&\quad c6 + \tfrac{1}{2} * a6 * x5 + \tfrac{1}{2} * b6 * y5 \neq 0 \lor c6 + \tfrac{1}{2} * a6 * r + \tfrac{1}{2} * a6 * x2 + \tfrac{1}{2} * b6 * y2 \neq 0 \lor \\
&\quad c7 + \tfrac{1}{2} * a7 * x9 + \tfrac{1}{2} * b7 * y9 \neq 0 \lor c7 + \tfrac{1}{2} * a7 * \mu3 + \tfrac{1}{2} * b7 * \mu4 \neq 0 \lor \\
&\quad c7 + \tfrac{1}{2} * a7 * x10 + \tfrac{1}{2} * a7 * x8 + \tfrac{1}{2} * b7 * y10 + \tfrac{1}{2} * b7 * y8 \neq 0 \lor \\
&\quad c7 + \tfrac{1}{2} * a7 * x5 + \tfrac{1}{2} * a7 * x6 + \tfrac{1}{2} * b7 * y5 + \tfrac{1}{2} * b7 * y6 \neq 0 \lor \\
&\quad c7 + \tfrac{1}{2} * a7 * x5 + \tfrac{1}{2} * b7 * y5 + \tfrac{1}{2} * a7 * \mu1 + \tfrac{1}{2} * b7 * \mu2 \neq 0 \lor \\
&\quad c8 + a8 * x3 + b8 * y3 \neq 0 \lor c8 + a8 * \mu1 + b8 * \mu2 \neq 0 \lor a5 * r \neq 0)
\end{aligned}$$

We now prove the above individual problems separately:

Proof of (Formula (TrisectingAngle): (1).1):

This proof problem has the following structure:

(Formula (TrisectingAngle): (1).1.structure)

$$\forall_{a5,a6,a7,a8,b5,b6,b7,b8,c5,c6,c7,c8,r,x1,x10,x2,x3,x4,x5,x6,x7,x8,x9,y1,y10,y2,y3,y4,y5,y6,y7,y8,y9,\eta1,\eta2,\eta3,\eta4,\eta5,\mu} (\mathrm{Poly}[1] \neq 0 \lor \mathrm{Poly}[2] \neq 0 \lor \mathrm{Poly}[3] \neq 0 \lor \mathrm{Poly}[4] \neq 0 \lor \mathrm{Poly}[5] \neq 0 \lor \mathrm{Poly}[6] \neq 0 \lor \mathrm{Poly}[7] \neq 0 \lor \mathrm{Poly}[8] \neq 0 \lor \mathrm{Poly}[9] \neq 0 \lor \mathrm{Poly}[10] \neq 0 \lor \mathrm{Poly}[11] \neq 0 \lor \mathrm{Poly}[12] \neq 0 \lor \mathrm{Poly}[13] \neq 0 \lor \mathrm{Poly}[14] \neq 0 \lor \mathrm{Poly}[15] \neq 0 \lor \mathrm{Poly}[16] \neq 0 \lor \mathrm{Poly}[17] \neq 0 \lor \mathrm{Poly}[18] \neq 0 \lor \mathrm{Poly}[19] \neq 0 \lor \mathrm{Poly}[20] \neq 0 \lor \mathrm{Poly}[21] \neq 0 \lor \mathrm{Poly}[22] \neq 0 \lor \mathrm{Poly}[23] \neq 0 \lor \mathrm{Poly}[24] \neq 0 \lor \mathrm{Poly}[25] \neq 0 \lor \mathrm{Poly}[26] \neq 0 \lor \mathrm{Poly}[27] \neq 0 \lor \mathrm{Poly}[28] \neq 0 \lor \mathrm{Poly}[29] \neq 0 \lor \mathrm{Poly}[30] \neq 0 \lor \mathrm{Poly}[31] \neq 0 \lor \mathrm{Poly}[32] \neq 0 \lor \mathrm{Poly}[33] \neq 0 \lor \mathrm{Poly}[34] \neq 0 \lor \mathrm{Poly}[35] \neq 0 \lor (\mathrm{Poly}[36] = 0))$$

,

where

$Poly[1] = c8$
$Poly[2] = x5$
$Poly[3] = x8$
$Poly[4] = -1 + r * \eta 1$
$Poly[5] = c5 + \frac{1}{2} * b5 * r$
$Poly[6] = (-r) + x4$
$Poly[7] = (-r) + x7$
$Poly[8] = (-b6 * x5) + a6 * y5$
$Poly[9] = (-b7 * x9) + a7 * y9$
$Poly[10] = (-b7 * \mu 3) + a7 * \mu 4$
$Poly[11] = -1 + a5^2 * \eta 2 + b5^2 * \eta 2$
$Poly[12] = -1 + a6^2 * \eta 3 + b6^2 * \eta 3$
$Poly[13] = -1 + a7^2 * \eta 4 + b7^2 * \eta 4$
$Poly[14] = -1 + a8^2 * \eta 5 + b8^2 * \eta 5$
$Poly[15] = c5 + a5 * x4 + b5 * y4$
$Poly[16] = c5 + a5 * x5 + b5 * y5$
$Poly[17] = c6 + a6 * x7 + b6 * y7$
$Poly[18] = c6 + a6 * x8 + b6 * y8$
$Poly[19] = c6 + a6 * \mu 3 + b6 * \mu 4$
$Poly[20] = c6 + \frac{1}{2} * a6 * x5 + \frac{1}{2} * b6 * y5$
$Poly[21] = c7 + \frac{1}{2} * a7 * x9 + \frac{1}{2} * b7 * y9$
$Poly[22] = c7 + \frac{1}{2} * a7 * \mu 3 + \frac{1}{2} * b7 * \mu 4$
$Poly[23] = c8 + a8 * x3 + b8 * y3$
$Poly[24] = c8 + a8 * \mu 1 + b8 * \mu 2$
$Poly[25] = b5 * r + (-b5 * x1) + a5 * y1$
$Poly[26] = b6 * r + (-b6 * x2) + a6 * y2$
$Poly[27] = c5 + \frac{1}{2} * a5 * r + \frac{1}{2} * a5 * x1 + \frac{1}{2} * b5 * y1$
$Poly[28] = c6 + \frac{1}{2} * a6 * r + \frac{1}{2} * a6 * x2 + \frac{1}{2} * b6 * y2$
$Poly[29] = (-b7 * x10) + b7 * x8 + a7 * y10 + (-a7 * y8)$
$Poly[30] = b7 * x5 + (-a7 * y5) + (-b7 * \mu 1) + a7 * \mu 2$
$Poly[31] = b7 * x5 + (-b7 * x6) + (-a7 * y5) + a7 * y6$
$Poly[32] = c7 + \frac{1}{2} * a7 * x10 + \frac{1}{2} * a7 * x8 + \frac{1}{2} * b7 * y10 + \frac{1}{2} * b7 * y8$
$Poly[33] = c7 + \frac{1}{2} * a7 * x5 + \frac{1}{2} * a7 * x6 + \frac{1}{2} * b7 * y5 + \frac{1}{2} * b7 * y6$
$Poly[34] = c7 + \frac{1}{2} * a7 * x5 + \frac{1}{2} * b7 * y5 + \frac{1}{2} * a7 * \mu 1 + \frac{1}{2} * b7 * \mu 2$
$Poly[35] = a5 * r$
$Poly[36] = x9^2 * y10 + (-2) * x10 * x9 * y9 + (-y10 * y9^2)$

(Formula (TrisectingAngle): (1).1.structure) is equivalent to

(Formula (TrisectingAngle): (1).1.implication)

$$\underset{a5,a6,a7,a8,b5,b6,b7,b8,c5,c6,c7,c8,r,x1,x10,x2,x3,x4,x5,x6,x7,x8,x9,y1,y10,y2,y3,y4,y5,y6,y7,y8,y9,\eta1,\eta2,\eta3,\eta4,\eta5,\mu}{\forall} ((\mathrm{Poly}[1]=0)\wedge(\mathrm{Poly}[2]=0)\wedge(\mathrm{Poly}[3]=0)\wedge(\mathrm{Poly}[4]=0)\wedge(\mathrm{Poly}[5]=0)\wedge (\mathrm{Poly}[6]=0)\wedge(\mathrm{Poly}[7]=0)\wedge(\mathrm{Poly}[8]=0)\wedge(\mathrm{Poly}[9]=0)\wedge(\mathrm{Poly}[10]=0)\wedge (\mathrm{Poly}[11]=0)\wedge(\mathrm{Poly}[12]=0)\wedge(\mathrm{Poly}[13]=0)\wedge(\mathrm{Poly}[14]=0)\wedge(\mathrm{Poly}[15]=0)\wedge (\mathrm{Poly}[16]=0)\wedge(\mathrm{Poly}[17]=0)\wedge(\mathrm{Poly}[18]=0)\wedge(\mathrm{Poly}[19]=0)\wedge(\mathrm{Poly}[20]=0)\wedge (\mathrm{Poly}[21]=0)\wedge(\mathrm{Poly}[22]=0)\wedge(\mathrm{Poly}[23]=0)\wedge(\mathrm{Poly}[24]=0)\wedge (\mathrm{Poly}[25]=0)\wedge(\mathrm{Poly}[26]=0)\wedge(\mathrm{Poly}[27]=0)\wedge(\mathrm{Poly}[28]=0)\wedge (\mathrm{Poly}[29]=0)\wedge(\mathrm{Poly}[30]=0)\wedge(\mathrm{Poly}[31]=0)\wedge(\mathrm{Poly}[32]=0)\wedge (\mathrm{Poly}[33]=0)\wedge(\mathrm{Poly}[34]=0)\wedge(\mathrm{Poly}[35]=0)\Rightarrow(\mathrm{Poly}[36]=0))$$

.

(Formula (TrisectingAngle): (1).1.implication) is equivalent to

(Formula (TrisectingAngle): (1).1.not-exists)

$$\underset{a5,a6,a7,a8,b5,b6,b7,b8,c5,c6,c7,c8,r,x1,x10,x2,x3,x4,x5,x6,x7,x8,x9,y1,y10,y2,y3,y4,y5,y6,y7,y8,y9,\eta1,\eta2,\eta3,\eta4,\eta5,\mu}{\nexists} (((\mathrm{Poly}[1]=0)\wedge(\mathrm{Poly}[2]=0)\wedge(\mathrm{Poly}[3]=0)\wedge(\mathrm{Poly}[4]=0)\wedge(\mathrm{Poly}[5]=0)\wedge (\mathrm{Poly}[6]=0)\wedge(\mathrm{Poly}[7]=0)\wedge(\mathrm{Poly}[8]=0)\wedge(\mathrm{Poly}[9]=0)\wedge(\mathrm{Poly}[10]=0)\wedge (\mathrm{Poly}[11]=0)\wedge(\mathrm{Poly}[12]=0)\wedge(\mathrm{Poly}[13]=0)\wedge(\mathrm{Poly}[14]=0)\wedge(\mathrm{Poly}[15]=0)\wedge (\mathrm{Poly}[16]=0)\wedge(\mathrm{Poly}[17]=0)\wedge(\mathrm{Poly}[18]=0)\wedge(\mathrm{Poly}[19]=0)\wedge (\mathrm{Poly}[20]=0)\wedge(\mathrm{Poly}[21]=0)\wedge(\mathrm{Poly}[22]=0)\wedge(\mathrm{Poly}[23]=0)\wedge (\mathrm{Poly}[24]=0)\wedge(\mathrm{Poly}[25]=0)\wedge(\mathrm{Poly}[26]=0)\wedge(\mathrm{Poly}[27]=0)\wedge (\mathrm{Poly}[28]=0)\wedge(\mathrm{Poly}[29]=0)\wedge(\mathrm{Poly}[30]=0)\wedge(\mathrm{Poly}[31]=0)\wedge (\mathrm{Poly}[32]=0)\wedge(\mathrm{Poly}[33]=0)\wedge(\mathrm{Poly}[34]=0)\wedge(\mathrm{Poly}[35]=0))\wedge\mathrm{Poly}[36]\neq 0)$$

.

By introducing the slack variable(s)

$\{\xi\}$

(Formula (TrisectingAngle): (1).1.not-exists) is transformed into the equivalent formula

(Formula (TrisectingAngle): (1).1.not-exists-slack)

$$\nexists_{a5,a6,a7,a8,b5,b6,b7,b8,c5,c6,c7,c8,r,x1,x10,x2,x3,x4,x5,x6,x7,x8,x9,y1,y10,y2,y3,y4,y5,y6,y7,y8,y9,\eta1,\eta2,\eta3,\eta4,\eta5,\mu}$$

$$(((\text{Poly}[1]=0)\land(\text{Poly}[2]=0)\land(\text{Poly}[3]=0)\land(\text{Poly}[4]=0)\land(\text{Poly}[5]=0)\land (\text{Poly}[6]=0)\land(\text{Poly}[7]=0)\land(\text{Poly}[8]=0)\land(\text{Poly}[9]=0)\land(\text{Poly}[10]=0)\land (\text{Poly}[11]=0)\land(\text{Poly}[12]=0)\land(\text{Poly}[13]=0)\land(\text{Poly}[14]=0)\land(\text{Poly}[15]=0)\land (\text{Poly}[16]=0)\land(\text{Poly}[17]=0)\land(\text{Poly}[18]=0)\land(\text{Poly}[19]=0)\land(\text{Poly}[20]=0)\land (\text{Poly}[21]=0)\land(\text{Poly}[22]=0)\land(\text{Poly}[23]=0)\land(\text{Poly}[24]=0)\land (\text{Poly}[25]=0)\land(\text{Poly}[26]=0)\land(\text{Poly}[27]=0)\land(\text{Poly}[28]=0)\land (\text{Poly}[29]=0)\land(\text{Poly}[30]=0)\land(\text{Poly}[31]=0)\land(\text{Poly}[32]=0)\land (\text{Poly}[33]=0)\land(\text{Poly}[34]=0)\land(\text{Poly}[35]=0))\land(-1+\xi\,\text{Poly}[36]=0))$$

.

Hence, we see that the proof problem is transformed into the question on whether or not a system of polynomial equations has a solution or not. This question can be answered by checking whether or not the (reduced) Groebner basis of

$$\{\text{Poly}[1], \text{Poly}[2], \text{Poly}[3], \text{Poly}[4], \text{Poly}[5], \text{Poly}[6], \text{Poly}[7], \text{Poly}[8], \text{Poly}[9], \text{Poly}[10], \text{Poly}[11], \text{Poly}[12], \text{Poly}[13], \text{Poly}[14], \text{Poly}[15], \text{Poly}[16], \text{Poly}[17], \text{Poly}[18], \text{Poly}[19], \text{Poly}[20], \text{Poly}[21], \text{Poly}[22], \text{Poly}[23], \text{Poly}[24], \text{Poly}[25], \text{Poly}[26], \text{Poly}[27], \text{Poly}[28], \text{Poly}[29], \text{Poly}[30], \text{Poly}[31], \text{Poly}[32], \text{Poly}[33], \text{Poly}[34], \text{Poly}[35], -1+\xi\,\text{Poly}[36]\}$$

is exactly {1}.

Hence, we compute the Groebner basis for the following polynomial list:

$$\{-1+x9^2\,y10\,\xi+(-2)\,x10\,x9\,y9\,\xi+(-1)\,y10\,y9^2\,\xi,\ c8,\ x5,\ x8,\ -1+r\,\eta1,\ -1+a5^2\,\eta2+b5^2\,\eta2,\ -1+a6^2\,\eta3+b6^2\,\eta3,\ -1+a7^2\,\eta4+b7^2\,\eta4,\ -1+a8^2\,\eta5+b8^2\,\eta5,\ c5+\frac{b5\,r}{2},\ -r+x4,\ -r+x7,\ b5\,r+(-1)\,b5\,x1+a5\,y1,\ b6\,r+(-1)\,b6\,x2+a6\,y2,\ b7\,x5+(-1)\,b7\,x6+(-1)\,a7\,y5+a7\,y6,\ b7\,x5+(-1)\,a7\,y5+(-1)\,b7\,\mu1+a7\,\mu2,\ -b7\,x10+b7\,x8+a7\,y10+(-1)\,a7\,y8,\ -b6\,x5+a6\,y5,\ -b7\,x9+a7\,y9,\ -b7\,\mu3+a7\,\mu4,\ c5+a5\,x4+b5\,y4,\ c5+a5\,x5+b5\,y5,\ c5+\frac{a5\,r}{2}+\frac{a5\,x1}{2}+\frac{b5\,y1}{2},\ c6+a6\,x7+b6\,y7,\ c6+a6\,x8+b6\,y8,\ c6+a6\,\mu3+b6\,\mu4,\ c6+\frac{a6\,x5}{2}+\frac{b6\,y5}{2},\ c6+\frac{a6\,r}{2}+\frac{a6\,x2}{2}+\frac{b6\,y2}{2},\ c7+\frac{a7\,x9}{2}+\frac{b7\,y9}{2},\ c7+\frac{a7\,\mu3}{2}+\frac{b7\,\mu4}{2},\ c7+\frac{a7\,x10}{2}+\frac{a7\,x8}{2}+\frac{b7\,y10}{2}+\frac{b7\,y8}{2},\ c7+\frac{a7\,x5}{2}+\frac{a7\,x6}{2}+\frac{b7\,y5}{2}+\frac{b7\,y6}{2},\ c7+\frac{a7\,x5}{2}+\frac{b7\,y5}{2}+\frac{a7\,\mu1}{2}+\frac{b7\,\mu2}{2},\ c8+a8\,x3+b8\,y3,\ c8+a8\,\mu1+b8\,\mu2,\ a5\,r\}$$

The Groebner basis:

$\{1\}$

Hence, (Formula (TrisectingAngle): (1).1) is proved.

Proof of (Formula (TrisectingAngle): (1).2):

This proof problem has the following structure:

(Formula (TrisectingAngle): (1).2.structure)

$$\underset{a5,a6,a7,a8,b5,b6,b7,b8,c5,c6,c7,c8,r,x1,x10,x2,x3,x4,x5,x6,x7,x8,x9,y1,y10,y2,y3,y4,y5,y6,y7,y8,y9,\eta1,\eta2,\eta3,\eta4,\eta5,\mu}{\forall} (\text{Poly}[1] \neq 0 \lor \text{Poly}[2] \neq 0 \lor \text{Poly}[3] \neq 0 \lor \text{Poly}[4] \neq 0 \lor \text{Poly}[5] \neq 0 \lor \text{Poly}[6] \neq 0 \lor \text{Poly}[7] \neq 0 \lor \text{Poly}[8] \neq 0 \lor \text{Poly}[9] \neq 0 \lor \text{Poly}[10] \neq 0 \lor \text{Poly}[11] \neq 0 \lor \text{Poly}[12] \neq 0 \lor \text{Poly}[13] \neq 0 \lor \text{Poly}[14] \neq 0 \lor \text{Poly}[15] \neq 0 \lor \text{Poly}[16] \neq 0 \lor \text{Poly}[17] \neq 0 \lor \text{Poly}[18] \neq 0 \lor \text{Poly}[19] \neq 0 \lor \text{Poly}[20] \neq 0 \lor \text{Poly}[21] \neq 0 \lor \text{Poly}[22] \neq 0 \lor \text{Poly}[23] \neq 0 \lor \text{Poly}[24] \neq 0 \lor \text{Poly}[25] \neq 0 \lor \text{Poly}[26] \neq 0 \lor \text{Poly}[27] \neq 0 \lor \text{Poly}[28] \neq 0 \lor \text{Poly}[29] \neq 0 \lor \text{Poly}[30] \neq 0 \lor \text{Poly}[31] \neq 0 \lor \text{Poly}[32] \neq 0 \lor \text{Poly}[33] \neq 0 \lor \text{Poly}[34] \neq 0 \lor \text{Poly}[35] \neq 0 \lor (\text{Poly}[36] = 0))$$

,

where

$$\begin{aligned}
\text{Poly}[1] &= c8\\
\text{Poly}[2] &= x5\\
\text{Poly}[3] &= x8\\
\text{Poly}[4] &= -1 + r * \eta 1\\
\text{Poly}[5] &= c5 + \tfrac{1}{2} * b5 * r\\
\text{Poly}[6] &= (-r) + x4\\
\text{Poly}[7] &= (-r) + x7\\
\text{Poly}[8] &= (-b6 * x5) + a6 * y5\\
\text{Poly}[9] &= (-b7 * x9) + a7 * y9\\
\text{Poly}[10] &= (-b7 * \mu 3) + a7 * \mu 4\\
\text{Poly}[11] &= -1 + a5^2 * \eta 2 + b5^2 * \eta 2\\
\text{Poly}[12] &= -1 + a6^2 * \eta 3 + b6^2 * \eta 3\\
\text{Poly}[13] &= -1 + a7^2 * \eta 4 + b7^2 * \eta 4\\
\text{Poly}[14] &= -1 + a8^2 * \eta 5 + b8^2 * \eta 5\\
\text{Poly}[15] &= c5 + a5 * x4 + b5 * y4\\
\text{Poly}[16] &= c5 + a5 * x5 + b5 * y5\\
\text{Poly}[17] &= c6 + a6 * x7 + b6 * y7\\
\text{Poly}[18] &= c6 + a6 * x8 + b6 * y8\\
\text{Poly}[19] &= c6 + a6 * \mu 3 + b6 * \mu 4\\
\text{Poly}[20] &= c6 + \tfrac{1}{2} * a6 * x5 + \tfrac{1}{2} * b6 * y5\\
\text{Poly}[21] &= c7 + \tfrac{1}{2} * a7 * x9 + \tfrac{1}{2} * b7 * y9\\
\text{Poly}[22] &= c7 + \tfrac{1}{2} * a7 * \mu 3 + \tfrac{1}{2} * b7 * \mu 4\\
\text{Poly}[23] &= c8 + a8 * x3 + b8 * y3\\
\text{Poly}[24] &= c8 + a8 * \mu 1 + b8 * \mu 2\\
\text{Poly}[25] &= b5 * r + (-b5 * x1) + a5 * y1\\
\text{Poly}[26] &= b6 * r + (-b6 * x2) + a6 * y2\\
\text{Poly}[27] &= c5 + \tfrac{1}{2} * a5 * r + \tfrac{1}{2} * a5 * x1 + \tfrac{1}{2} * b5 * y1\\
\text{Poly}[28] &= c6 + \tfrac{1}{2} * a6 * r + \tfrac{1}{2} * a6 * x2 + \tfrac{1}{2} * b6 * y2\\
\text{Poly}[29] &= (-b7 * x10) + b7 * x8 + a7 * y10 + (-a7 * y8)\\
\text{Poly}[30] &= b7 * x5 + (-a7 * y5) + (-b7 * \mu 1) + a7 * \mu 2\\
\text{Poly}[31] &= b7 * x5 + (-b7 * x6) + (-a7 * y5) + a7 * y6\\
\text{Poly}[32] &= c7 + \tfrac{1}{2} * a7 * x10 + \tfrac{1}{2} * a7 * x8 + \tfrac{1}{2} * b7 * y10 + \tfrac{1}{2} * b7 * y8\\
\text{Poly}[33] &= c7 + \tfrac{1}{2} * a7 * x5 + \tfrac{1}{2} * a7 * x6 + \tfrac{1}{2} * b7 * y5 + \tfrac{1}{2} * b7 * y6\\
\text{Poly}[34] &= c7 + \tfrac{1}{2} * a7 * x5 + \tfrac{1}{2} * b7 * y5 + \tfrac{1}{2} * a7 * \mu 1 + \tfrac{1}{2} * b7 * \mu 2\\
\text{Poly}[35] &= a5 * r\\
\text{Poly}[36] &= x9^3 * y3 + (-3) * x3 * x9^2 * y9 + (-3) * x9 * y3 * y9^2 + x3 * y9^3
\end{aligned}$$

(Formula (TrisectingAngle): (1).2.structure) is equivalent to

(Formula (TrisectingAngle): (1).2.implication)

$$\underset{a5,a6,a7,a8,b5,b6,b7,b8,c5,c6,c7,c8,r,x1,x10,x2,x3,x4,x5,x6,x7,x8,x9,y1,y10,y2,y3,y4,y5,y6,y7,y8,y9,\eta1,\eta2,\eta3,\eta4,\eta5,\mu}{\forall} ((\text{Poly}[1]=0)\wedge(\text{Poly}[2]=0)\wedge(\text{Poly}[3]=0)\wedge(\text{Poly}[4]=0)\wedge(\text{Poly}[5]=0)\wedge(\text{Poly}[6]=0)\wedge(\text{Poly}[7]=0)\wedge(\text{Poly}[8]=0)\wedge(\text{Poly}[9]=0)\wedge(\text{Poly}[10]=0)\wedge(\text{Poly}[11]=0)\wedge(\text{Poly}[12]=0)\wedge(\text{Poly}[13]=0)\wedge(\text{Poly}[14]=0)\wedge(\text{Poly}[15]=0)\wedge(\text{Poly}[16]=0)\wedge(\text{Poly}[17]=0)\wedge(\text{Poly}[18]=0)\wedge(\text{Poly}[19]=0)\wedge(\text{Poly}[20]=0)\wedge(\text{Poly}[21]=0)\wedge(\text{Poly}[22]=0)\wedge(\text{Poly}[23]=0)\wedge(\text{Poly}[24]=0)\wedge(\text{Poly}[25]=0)\wedge(\text{Poly}[26]=0)\wedge(\text{Poly}[27]=0)\wedge(\text{Poly}[28]=0)\wedge(\text{Poly}[29]=0)\wedge(\text{Poly}[30]=0)\wedge(\text{Poly}[31]=0)\wedge(\text{Poly}[32]=0)\wedge(\text{Poly}[33]=0)\wedge(\text{Poly}[34]=0)\wedge(\text{Poly}[35]=0)\Rightarrow(\text{Poly}[36]=0))$$

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(Formula (TrisectingAngle): (1).2.implication) is equivalent to

(Formula (TrisectingAngle): (1).2.not-exists)

$$\underset{a5,a6,a7,a8,b5,b6,b7,b8,c5,c6,c7,c8,r,x1,x10,x2,x3,x4,x5,x6,x7,x8,x9,y1,y10,y2,y3,y4,y5,y6,y7,y8,y9,\eta1,\eta2,\eta3,\eta4,\eta5,\mu}{\nexists} (((\text{Poly}[1]=0)\wedge(\text{Poly}[2]=0)\wedge(\text{Poly}[3]=0)\wedge(\text{Poly}[4]=0)\wedge(\text{Poly}[5]=0)\wedge(\text{Poly}[6]=0)\wedge(\text{Poly}[7]=0)\wedge(\text{Poly}[8]=0)\wedge(\text{Poly}[9]=0)\wedge(\text{Poly}[10]=0)\wedge(\text{Poly}[11]=0)\wedge(\text{Poly}[12]=0)\wedge(\text{Poly}[13]=0)\wedge(\text{Poly}[14]=0)\wedge(\text{Poly}[15]=0)\wedge(\text{Poly}[16]=0)\wedge(\text{Poly}[17]=0)\wedge(\text{Poly}[18]=0)\wedge(\text{Poly}[19]=0)\wedge(\text{Poly}[20]=0)\wedge(\text{Poly}[21]=0)\wedge(\text{Poly}[22]=0)\wedge(\text{Poly}[23]=0)\wedge(\text{Poly}[24]=0)\wedge(\text{Poly}[25]=0)\wedge(\text{Poly}[26]=0)\wedge(\text{Poly}[27]=0)\wedge(\text{Poly}[28]=0)\wedge(\text{Poly}[29]=0)\wedge(\text{Poly}[30]=0)\wedge(\text{Poly}[31]=0)\wedge(\text{Poly}[32]=0)\wedge(\text{Poly}[33]=0)\wedge(\text{Poly}[34]=0)\wedge(\text{Poly}[35]=0))\wedge\text{Poly}[36]\neq 0)$$

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By introducing the slack variable(s)

$\{\xi 1\}$

(Formula (TrisectingAngle): (1).2.not-exists) is transformed into the equivalent formula

(Formula (TrisectingAngle): (1).2.not-exists-slack)

$$\nexists_{a5,a6,a7,a8,b5,b6,b7,b8,c5,c6,c7,c8,r,x1,x10,x2,x3,x4,x5,x6,x7,x8,x9,y1,y10,y2,y3,y4,y5,y6,y7,y8,y9,\eta1,\eta2,\eta3,\eta4,\eta5,\mu}$$

$$(((\mathrm{Poly}[1]=0)\wedge(\mathrm{Poly}[2]=0)\wedge(\mathrm{Poly}[3]=0)\wedge(\mathrm{Poly}[4]=0)\wedge(\mathrm{Poly}[5]=0)\wedge(\mathrm{Poly}[6]=0)\wedge(\mathrm{Poly}[7]=0)\wedge(\mathrm{Poly}[8]=0)\wedge(\mathrm{Poly}[9]=0)\wedge(\mathrm{Poly}[10]=0)\wedge(\mathrm{Poly}[11]=0)\wedge(\mathrm{Poly}[12]=0)\wedge(\mathrm{Poly}[13]=0)\wedge(\mathrm{Poly}[14]=0)\wedge(\mathrm{Poly}[15]=0)\wedge(\mathrm{Poly}[16]=0)\wedge(\mathrm{Poly}[17]=0)\wedge(\mathrm{Poly}[18]=0)\wedge(\mathrm{Poly}[19]=0)\wedge(\mathrm{Poly}[20]=0)\wedge(\mathrm{Poly}[21]=0)\wedge(\mathrm{Poly}[22]=0)\wedge(\mathrm{Poly}[23]=0)\wedge(\mathrm{Poly}[24]=0)\wedge(\mathrm{Poly}[25]=0)\wedge(\mathrm{Poly}[26]=0)\wedge(\mathrm{Poly}[27]=0)\wedge(\mathrm{Poly}[28]=0)\wedge(\mathrm{Poly}[29]=0)\wedge(\mathrm{Poly}[30]=0)\wedge(\mathrm{Poly}[31]=0)\wedge(\mathrm{Poly}[32]=0)\wedge(\mathrm{Poly}[33]=0)\wedge(\mathrm{Poly}[34]=0)\wedge(\mathrm{Poly}[35]=0))\wedge(-1+\xi1\,\mathrm{Poly}[36]=0))$$

.

Hence, we see that the proof problem is transformed into the question on whether or not a system of polynomial equations has a solution or not. This question can be answered by checking whether or not the (reduced) Groebner basis of

$$\{\mathrm{Poly}[1],\mathrm{Poly}[2],\mathrm{Poly}[3],\mathrm{Poly}[4],\mathrm{Poly}[5],\mathrm{Poly}[6],\mathrm{Poly}[7],\mathrm{Poly}[8],\mathrm{Poly}[9],\mathrm{Poly}[10],\mathrm{Poly}[11],\mathrm{Poly}[12],\mathrm{Poly}[13],\mathrm{Poly}[14],\mathrm{Poly}[15],\mathrm{Poly}[16],\mathrm{Poly}[17],\mathrm{Poly}[18],\mathrm{Poly}[19],\mathrm{Poly}[20],\mathrm{Poly}[21],\mathrm{Poly}[22],\mathrm{Poly}[23],\mathrm{Poly}[24],\mathrm{Poly}[25],\mathrm{Poly}[26],\mathrm{Poly}[27],\mathrm{Poly}[28],\mathrm{Poly}[29],\mathrm{Poly}[30],\mathrm{Poly}[31],\mathrm{Poly}[32],\mathrm{Poly}[33],\mathrm{Poly}[34],\mathrm{Poly}[35],-1+\xi1\,\mathrm{Poly}[36]\}$$

is exactly {1}.

Hence, we compute the Groebner basis for the following polynomial list:

$$\{-1+x9^3\,y3\,\xi1+(-3)\,x3\,x9^2\,y9\,\xi1+(-3)\,x9\,y3\,y9^2\,\xi1+x3\,y9^3\,\xi1,\ c8,\ x5,\ x8,\ -1+r\,\eta1,\ -1+a5^2\,\eta2+b5^2\,\eta2,\ -1+a6^2\,\eta3+b6^2\,\eta3,\ -1+a7^2\,\eta4+b7^2\,\eta4,\ -1+a8^2\,\eta5+b8^2\,\eta5,\ c5+\frac{b5\,r}{2},\ -r+x4,\ -r+x7,\ b5\,r+(-1)\,b5\,x1+a5\,y1,\ b6\,r+(-1)\,b6\,x2+a6\,y2,\ b7\,x5+(-1)\,b7\,x6+(-1)\,a7\,y5+a7\,y6,\ b7\,x5+(-1)\,a7\,y5+(-1)\,b7\,\mu1+a7\,\mu2,\ -b7\,x10+b7\,x8+a7\,y10+(-1)\,a7\,y8,\ -b6\,x5+a6\,y5,\ -b7\,x9+a7\,y9,\ -b7\,\mu3+a7\,\mu4,\ c5+a5\,x4+b5\,y4,\ c5+a5\,x5+b5\,y5,\ c5+\frac{a5\,r}{2}+\frac{a5\,x1}{2}+\frac{b5\,y1}{2},\ c6+a6\,x7+b6\,y7,\ c6+a6\,x8+b6\,y8,\ c6+a6\,\mu3+b6\,\mu4,\ c6+\frac{a6\,x5}{2}+\frac{b6\,y5}{2},\ c6+\frac{a6\,r}{2}+\frac{a6\,x2}{2}+\frac{b6\,y2}{2},\ c7+\frac{a7\,x9}{2}+\frac{b7\,y9}{2},\ c7+\frac{a7\,\mu3}{2}+\frac{b7\,\mu4}{2},\ c7+\frac{a7\,x10}{2}+\frac{a7\,x8}{2}+\frac{b7\,y10}{2}+\frac{b7\,y8}{2},\ c7+\frac{a7\,x5}{2}+\frac{a7\,x6}{2}+\frac{b7\,y5}{2}+\frac{b7\,y6}{2},\ c7+\frac{a7\,x5}{2}+\frac{b7\,y5}{2}+\frac{a7\,\mu1}{2}+\frac{b7\,\mu2}{2},\ c8+a8\,x3+b8\,y3,\ c8+a8\,\mu1+b8\,\mu2,\ a5\,r\}$$

The Groebner basis:

$$\{1\}$$

Hence, (Formula (TrisectingAngle): (1).2) is proved.

Since all of the individual subtheorems are proved, the original formula is proved.

□

■ Additional Proof Generation Information